Classical Electrodynamics Notes

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May 10, 2024 —— September 7, 2024

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Introduction and Tips

Editor to reader

The purpose of editing this book is to provide fancy notes for Classical Electrodynamics by Jackson. While the fact that there is actually a bit of difficulty for reading throughout Jackson's book, especially in some chapters at graduate level and even some sections containing much more professional mathematical methods with less math derivation process for undergraduates, I can promise that you will make your learning process much more convenient with this notebook. However, there is a premise that 1. you have a strong willing to learn the course, Electrodynamics, while the reason is not just completing your curriculum course, which means that you want to get a better understanding of it, rather just to have a higher grades, 2. you have a strong heart with which you can defeat any problem, including but not limited to mathematical methods and physical picture front of your learning road.

Emphasize again. Strong willing and strong heart.

Vector Operators: Grad, Div, Curl and Laplacian

Following are three imperative calculus methods, I'd like just mention and summarize key points of them rather than going deep inside and giving proof process, which must be contained in any good textbook you own.

Three field operators:

- the **gradient** of a scalar field
- the **divergence** of a vector field
- the **curl** of a vector field

There are two points to get over about each:

- The mechanics of taking the grad, div and curl, for which you will need to brush up your **multivariate calculus**.
- The underlying **physical meaning** — that is, why they are worth bothering about.

The gradient of a scalar field

If $U(\mathbf{r}) = U(x, y, z)$ is a scalar field, then its **gradient** at any point is defined in Cartesian coordinates by

$$grad U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \tag{0.2.1.1}$$

Now we can define the **vector operator** which is usually called "nabla"

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{0.2.1.2}$$

The interesting thing of gradient is that, the gradient of a scalar field tends to point in the direction of **greatest change** of the field.

It's important to have a physical view of it, after thinking a short time, we can have following two properties:

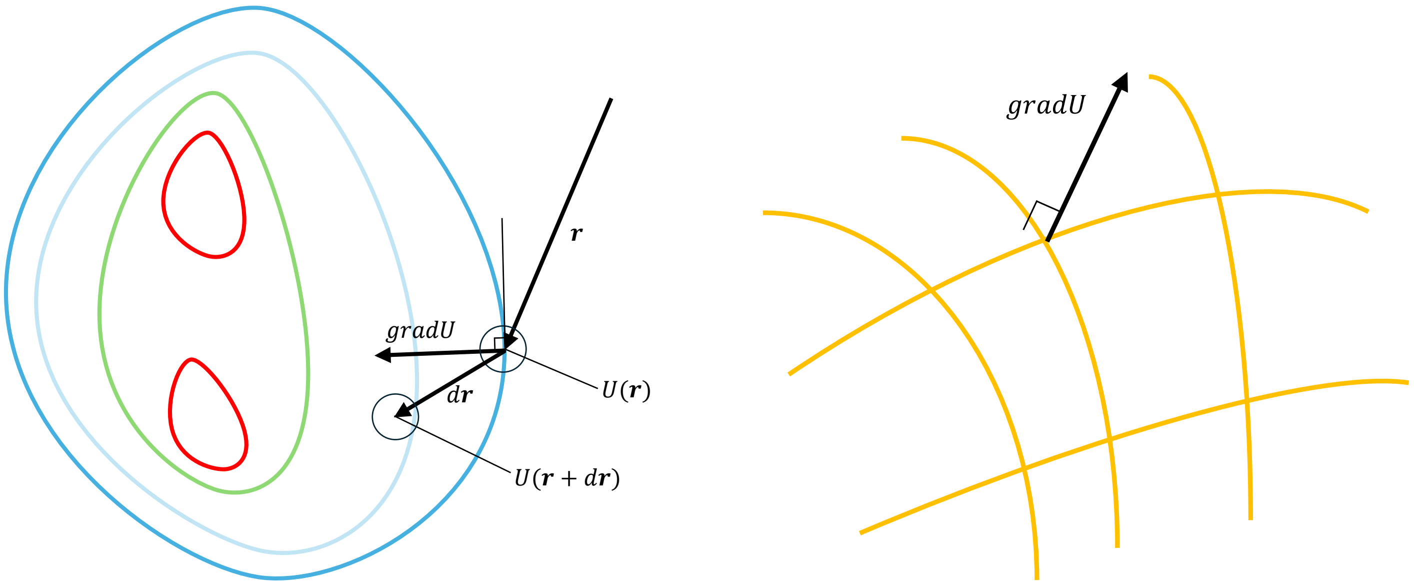
- directional derivative:

$$dU = \nabla U \cdot d\mathbf{r} \tag{0.2.1.3}$$

- normal identities:

$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 \tag{0.2.1.4}$$

  note that ds represents a tiny amount displacement **within** the surface, $U(\mathbf{r}) = constant$, which means that $\frac{d\mathbf{r}}{ds}$ is a tangent to the surface.



(a) The directional derivative

(b) Surface of constant U

Fig. 0.2.1.1. Two properties of gradient

The divergence of a vector field

If $\mathbf{a}(x, y, z)$ is a vector function of position in three dimensions, then its divergence at any point is defined in Cartesian coordinates by

$$div\mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \tag{0.2.2.1}$$

With the defination of vector operator ∇ in the previous section, we can write the above equation in a simplified notation:

$$div\mathbf{a} = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}) \cdot \mathbf{a} = \nabla \cdot \mathbf{a} \tag{0.2.2.2}$$

Notice that the divergence of a vector field is a scalar field.

To be specific, lets consider a volume element $dV = dxdydz$ in Cartesian coordinates, and firstly we just set our eyes on the face of area $dxdz$, which is perpendicular to the y axis and facing outwards in the negative y direction. (The one with surface area $d\mathbf{S} = -dxdz\hat{\mathbf{j}}$)

We can easily see its contribution to the **OUTWARD** flux from this surface is

$$\mathbf{a} \cdot d\mathbf{S} = -a_y(x, y, z)dzdx \tag{0.2.2.3}$$

Now you're expected to write down the contribution of opposite surface area. The amount is

$$a_y(x, y + dy, z)dzdx = (a_y + \frac{\partial a_y}{\partial y}dy)dxdz \tag{0.2.2.4}$$

Lets add them both and the total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y}dydxdz = \frac{\partial a_y}{\partial y}dV \tag{0.2.2.5}$$

Here we sum the other faces and give a total outward flux of the volume element dV

$$(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z})dV = \nabla \cdot \mathbf{a}\ dV \tag{0.2.2.6}$$

So we can see that

- The divergence of a vector field represents the flux generation per unit volume at each point of the field, and it means an efflux not an influx.

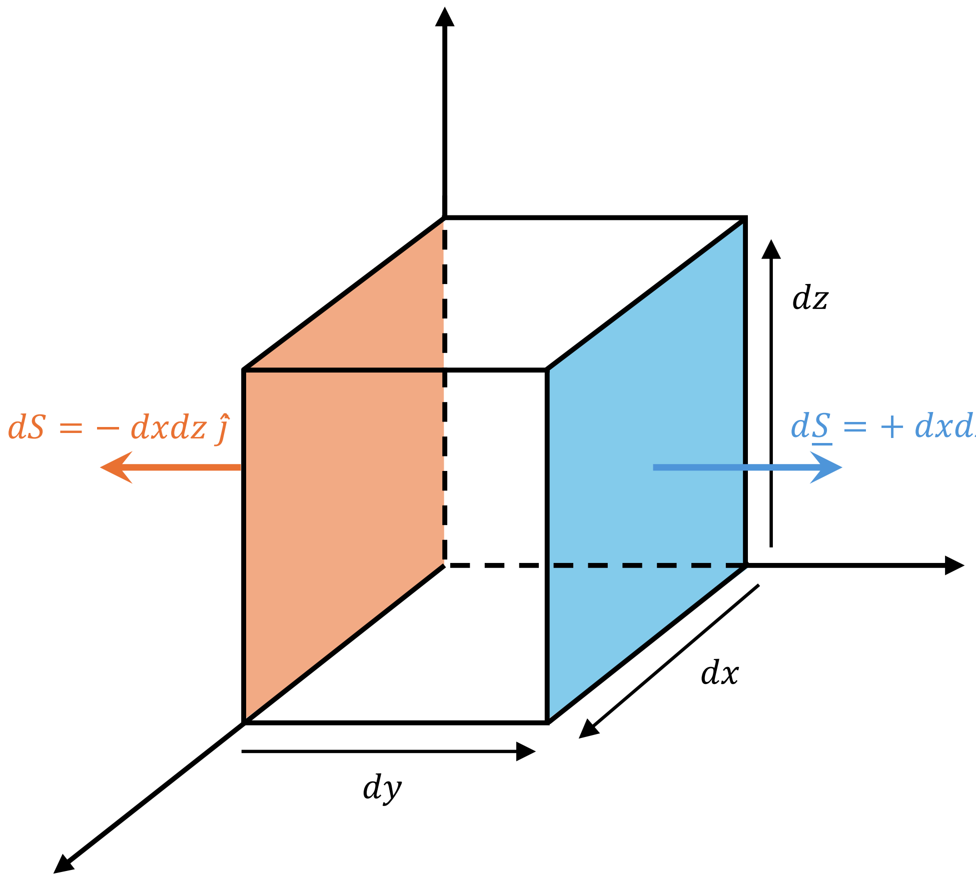


Fig. 0.2.2.1. Elemental volume for calculating divergence

The laplacian $div(gradU)$ of a scalar field

Recall the definition in previous sections, we can certainly compute $div(gradU)$.

Now and later, we may only use ∇ operator making derivation handy.

$$\nabla \cdot (\nabla U) = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}) \cdot ((\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z})U) \tag{0.2.3.1a}$$

$$= ((\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}) \cdot (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}))U \tag{0.2.3.1b}$$

$$= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})U \tag{0.2.3.1c}$$

$$= (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}) \tag{0.2.3.1d}$$

Now we can take a step to simplify the last expression, which occurs frequently in engineering science, as the operator ∇^2, which is also called the "Laplacian".

$$\nabla^2 U = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})U \tag{0.2.3.2}$$

Tip: usually we may apply the ∇^2 **exerting on the vector field**. The solution is as follows.

$$\nabla^2 \mathbf{a} = \nabla^2 a_x \hat{\mathbf{i}} + \nabla^2 a_y \hat{\mathbf{j}} + \nabla^2 a_z \hat{\mathbf{k}} \tag{0.2.3.3}$$

Here is a classical and important example for you to exercise, with the derivation process omitted.

$$\nabla^2 \frac{1}{r} = 0 \tag{0.2.3.4}$$

Notice that $r = \sqrt{x^2 + y^2 + z^2}$.

The curl of a vector field

We are now overwhelmed by an irrestible temptation to **cross it** with a vector field $\nabla \times \mathbf{a}$.

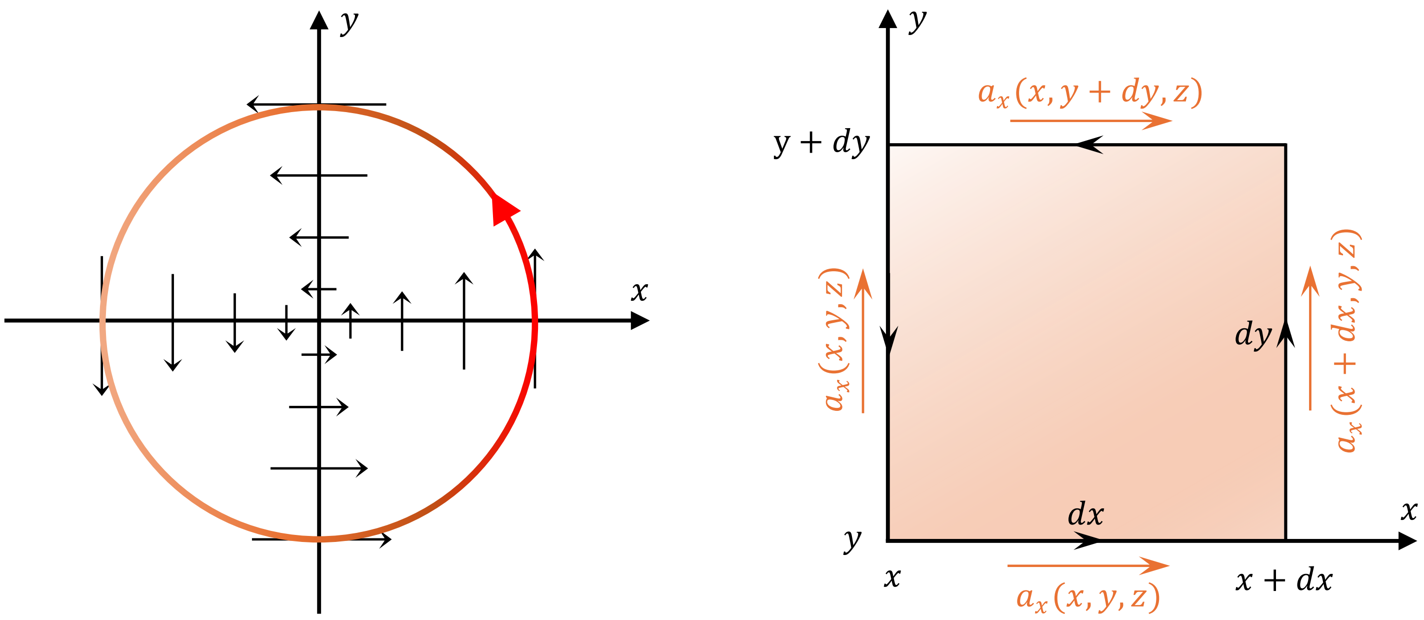
$$\nabla \times \mathbf{a} \equiv curl(\mathbf{a}) \tag{0.2.4.1}$$

And we can follow the pseudo-determinant recipe for vector products.

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \qquad (remember\ it\ this\ way) \tag{0.2.4.2a}$$

$$= (\frac{\partial a_z}{\partial x} - \frac{\partial a_y}{\partial z})\hat{\mathbf{i}} + (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x})\hat{\mathbf{j}} + (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y})\hat{\mathbf{k}} \tag{0.2.4.2b}$$

Perhaps the following example can give a brandnew view of curl. The field $\mathbf{a} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ is sketched in **Fig. 0.2.4.1.a**. You can also get that a field like this must give a finite value to the line integral around the complete loop $\oint \mathbf{a} \cdot d\mathbf{r}$.



(a) Rough sketch of example vector field **(b)** Small element for calculating curl

Fig. 0.2.4.1. Brandnew view of curl

In fact, curl is closely related to the line integral around the loop.

Here I'd like give a easy and rough proof, which will give you a good perspective of it. Consider the circulation round the perimeter of a rectangular element dx by dy shown in **Fig. 0.2.4.1.b**. The fields in the x direction **at the bottom and top** are

$$a_x(x, y, z) \qquad and \qquad a_x(x, y+dy, z) = a_x(x, y, z) + \frac{\partial a_x}{\partial y} dy \tag{0.2.4.3}$$

So as well, you can calculate the fields in the y direction, which are omitted. Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows.

$$dC = +[a_x \ dx] + [a_y(x+dx, y, z) \ dy] - [a_x(x, y+dy, z) \ dx] - [a_y \ dy] \tag{0.2.4.4a}$$

$$= +[a_x \ dx] + [(a_y + \frac{\partial a_y}{\partial x} dx) \ dy] - [(a_x + \frac{\partial a_x}{\partial y} dx) \ dx] - [a_y \ dy] \tag{0.2.4.4b}$$

$$= (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}) \ dx \ dy \tag{0.2.4.4c}$$

$$= (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \tag{0.2.4.4d}$$

where $d\mathbf{S} = dxdy\hat{\mathbf{k}}$.

Some definitions involving grad, div and curl

- A scalar field with zero gradient is said to be **constant**.
- A vector field with zero divergence is said to be **solenoidal**.
- A vector field with zero curl is said to be **irrotational**.

Vector Operator Identities

In this section, we move our sight on the next important part, looking at more complicated identities involving vector operators. Here is one main thing to appreciate, the operators, which are mentioned in previous sections, behave both as **vectors** and as **differential** operators, so that we must take care of the product with the usual rules of taking the derivative.

For giving a overview of identities, I'd like listing them first. Just be careful what are vector and scalar.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{0.3.1a}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{0.3.1b}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{0.3.1c}$$
$$\nabla\times\nabla\varphi = \mathbf{0} \tag{0.3.1d}$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0 \tag{0.3.1e}$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a} \tag{0.3.1f}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a} \tag{0.3.1g}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a} \tag{0.3.1h}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) \tag{0.3.1i}$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \tag{0.3.1j}$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} \tag{0.3.1k}$$

Some proofs of identities

Here's some brief derivations for giving proofs of some important identities, while the rest of them can be easily proved on your own.

- **curl grad** $U = \mathbf{0}$: $\nabla\times\nabla\varphi = \mathbf{0}$ (0.3.1d)

$$\nabla\times\nabla\varphi = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} \tag{0.3.1.1a}$$
$$= \hat{\mathbf{i}}(\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial^2\varphi}{\partial y\partial z}) + \hat{\mathbf{j}}(\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial z\partial x}) + \hat{\mathbf{k}}(\frac{\partial^2\varphi}{\partial y\partial x} - \frac{\partial^2\varphi}{\partial x\partial y}) \tag{0.3.1.1b}$$
$$= \mathbf{0} \tag{0.3.1.1c}$$

- **div curl a** $= 0$: $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ (0.3.1e)

$$\nabla \cdot (\nabla \times \mathbf{a}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad (0.3.1.2a)$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \quad (0.3.1.2b)$$

$$= 0 \quad (0.3.1.2c)$$

- **div of a** $\times$ **b**: $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ (0.3.1j)

This proof shows that it's a bit trickier when vector or scalar products involved.

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} & \frac{\partial}{\partial x_k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (0.3.1.3a)$$

$$= \frac{\partial}{\partial x}(a_y b_z - a_z b_y) + \frac{\partial}{\partial y}(a_z b_x - a_x b_z) + \frac{\partial}{\partial z}(a_x b_y - a_y b_x) \quad (0.3.1.3b)$$

$$= \ldots\ldots \qquad (Bash\ out\ the\ products)$$

$$= \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b} \quad (0.3.1.3c)$$

- **curl of a** $\times$ **b**: $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$ (0.3.1k)

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix} \quad (0.3.1.4)$$

Owing to the tricky derivations, We calculate the $\hat{\mathbf{i}}$ component first.

$$The\ \hat{\mathbf{i}}\ component\ is\ \frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z) \quad (0.3.1.5a)$$

$$= a_x(\frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}) - b_x(\frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}) + (b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z})a_x - (a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z})b_x \quad (0.3.1.5b)$$

Adding $a_x \frac{\partial b_x}{\partial x}$ to the first of these, and substracting it from the last, and doing the same with $b_x \frac{\partial a_x}{\partial x}$ to the other two terms, we get a equivalent form.

$$a_x(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}) - b_x(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}) + (b_x \frac{\partial}{\partial x} + b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z})a_x - (a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z})b_x \quad (0.3.1.6)$$

Suming all components as the case of $\hat{\mathbf{i}}$, we can find that:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \quad (0.3.1k)$$

- **curl curl a**: $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$ (0.3.1f)

  This derivation is one of the triciest of the list. We firstly handle the left-hand side:

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_x & a_y & a_z\end{vmatrix} \tag{0.3.1.7a}$$

$$= \nabla\times[(\frac{\partial a_z}{\partial x} - \frac{\partial a_y}{\partial z})\hat{\mathbf{i}} + (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x})\hat{\mathbf{j}} + (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y})\hat{\mathbf{k}}] \tag{0.3.1.7b}$$

$$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial a_z}{\partial x} - \frac{\partial a_y}{\partial z} & \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} & \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\end{vmatrix} \tag{0.3.1.7c}$$

$$\textit{The } \hat{\mathbf{i}} \textit{ component is } \frac{\partial}{\partial y}(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}) - \frac{\partial}{\partial z}(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}) \tag{0.3.1.8a}$$

$$= (\frac{\partial^2 a_y}{\partial x\partial y} - \frac{\partial^2 a_x}{\partial y^2}) - (\frac{\partial^2 a_x}{\partial z^2} - \frac{\partial^2 a_z}{\partial x\partial z}) \tag{0.3.1.8b}$$

$$= (\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_y}{\partial x\partial y} + \frac{\partial^2 a_z}{\partial x\partial z}) - (\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2}) \tag{0.3.1.8c}$$

  The first term on the right is the $\hat{\mathbf{i}}$ component of $\nabla(\nabla\cdot\mathbf{a})$, while the last term on the right is the $\hat{\mathbf{i}}$ component of $\nabla^2\mathbf{a}$.

  Suming all components as the case of $\hat{\mathbf{i}}$, we can find that:

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a} \tag{0.3.1f}$$

Definition of the operator $\mathbf{a}\cdot\nabla$

This is a scalar operator, but obviously it can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field.

$$\mathbf{a}\cdot\nabla \equiv a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z} \tag{0.3.2.1}$$

Curvilinear Coordinates

So far, we have learned the Cartesian coordinates, and likely the plane, cylindrical ,or spherical polars, which may be considered exotic. But often the symmetry of the problem strongly hints that we should use another coordinate system, the **curvilinear coordinate system**, which is expected to be more exotic and general.

The curvilinear coordinate system also has unit vectors. Though maybe we don't know how to define them easily like how we did to Cartesian coordinates, we can start with line integrals, as they raise all the issues and provide the simplest case. Later I will show you the benefits in the following section.

The length scales

When we perform a line integral, we set our eyes on the expression for d**r** as a sum of terms involving the basis vectors (unit vectors). In Cartesian coordinates, we write

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad \textit{and} \quad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{0.4.1.1}$$

Here is one thing interesting, we can use changes (dx, dy, dz) along the basis vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) because these are independent of each other. The Cartesian coordinate system is also **orthogonal**, which means any basis vectors are mutually orthogonal. The orthogonal coordinate is far more special than just the independence of basis vectors. So I must remind you that the general curvilinear coordinate system **doesn't have to be** orthogonal while we just want basis vectors to be independent. But in fact, we often and generally care about orthogonal curvilinear coordinates, due to the excellent mathematical and geometric properties of the orthogonal coordinate system. In this section, we assume that we are talking about the general curvilinear coordinates, unless we specify that it is an orthogonal curvilinear coordinates.

As we performing a line integral, in Cartesians, we have

$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{0.4.1.2}$$

It's time to perform it on curvilinear coordinate. Firstly we must define at least three basis vectors, how about $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$. But clearly we can't easily obtain an expression for $d\mathbf{r}$ as a sum of terms involving three new basis vectors, for the length scales are properly handled in Cartesians while generally may not in curvilinear coordinates. The key task is finding the lost **length scales**.

Let's brush up the multivariate calculus first, and for simplicity, let us think about a line integral in the plane, and transform from (x, y) to (u, v) coordinates.

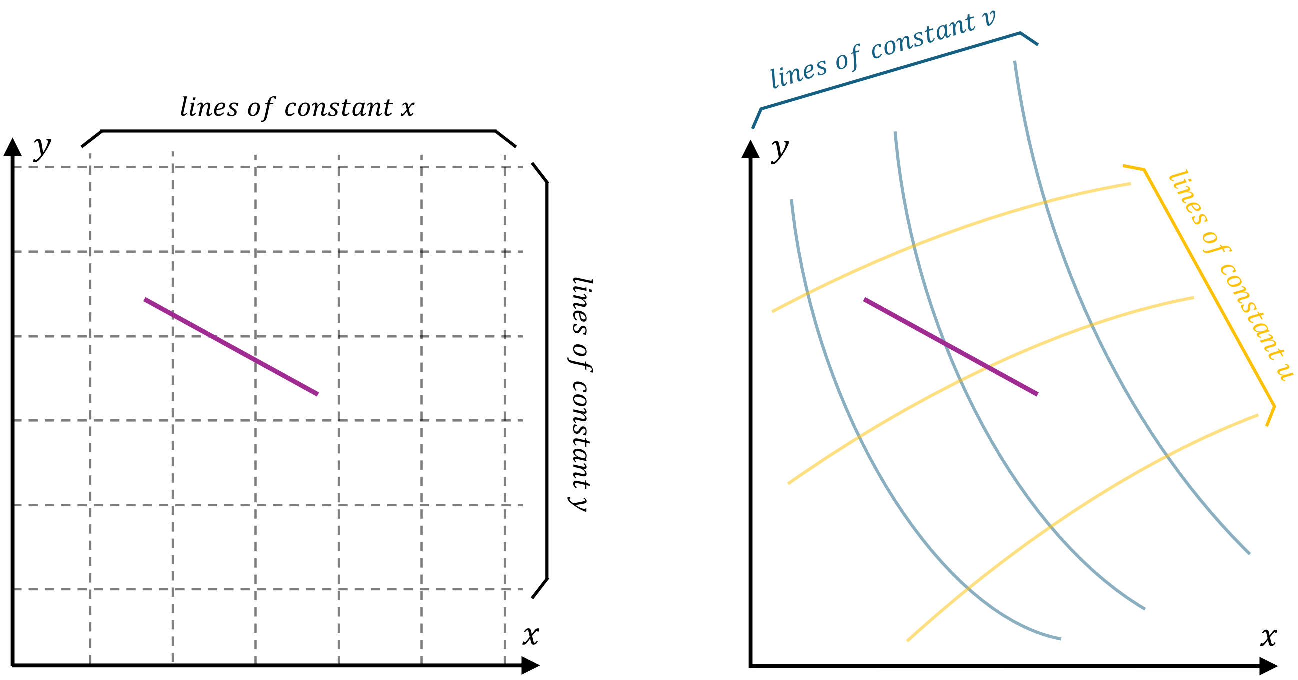


Fig. 0.4.1.1. Lines of constant u and v appear as curves on the xy-plane

Here we have

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \qquad and \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \tag{0.4.1.3}$$

Hence, as $\mathbf{r} = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}}$, we can write

$$d\mathbf{r} = (\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv)\hat{\mathbf{i}} + (\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv)\hat{\mathbf{j}} \tag{0.4.1.4a}$$

$$= (\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}})du + (\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}})dv \tag{0.4.1.4b}$$

$$= (h_u\hat{\mathbf{u}})du + (h_v\hat{\mathbf{v}})dv \tag{0.4.1.4c}$$

Easily we get the expresstions for $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and simultaneously get the length scales h_u and h_v. These scales are called **metric coefficients**. They are the factors that turn the "d-whatevers" into proper lengths.

Because $\hat{\mathbf{u}}$ is a unit vector, if we square both sides of the expression of $h_u\hat{\mathbf{u}}$ we find that

$$h_u = \left[(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2\right]^{\frac{1}{2}} \tag{0.4.1.5}$$

In curvilinear coordinates, we also have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv \tag{0.4.1.6}$$

Combining the **Eq. 0.4.1.4c** and **Eq. 0.4.1.6**, we also have

$$h_u\hat{\mathbf{u}} = \frac{\partial r}{\partial u} \qquad and \qquad h_u = |\frac{\partial r}{\partial u}| \tag{0.4.1.7}$$

and similarly for v.

The tangents

It's worthwhile discovering the tangent to the curve in curvilinear coordinates. Perhaps you have already know the tangents in Cartesians. For example, surpose we get a curve $y(x)$ in the plane $z = constant$, then we get that $\frac{dy}{dx}$ is a (non-unit) tangent to the curve $y(x)$. Now we want to write down the tangent to the $v = constant$ curve. We already have $\mathbf{r} = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}}$ and so we get

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}} \tag{0.4.2.1}$$

This is like $\frac{dy}{dx}$ but is partial because there are two parameters with v being held constant.

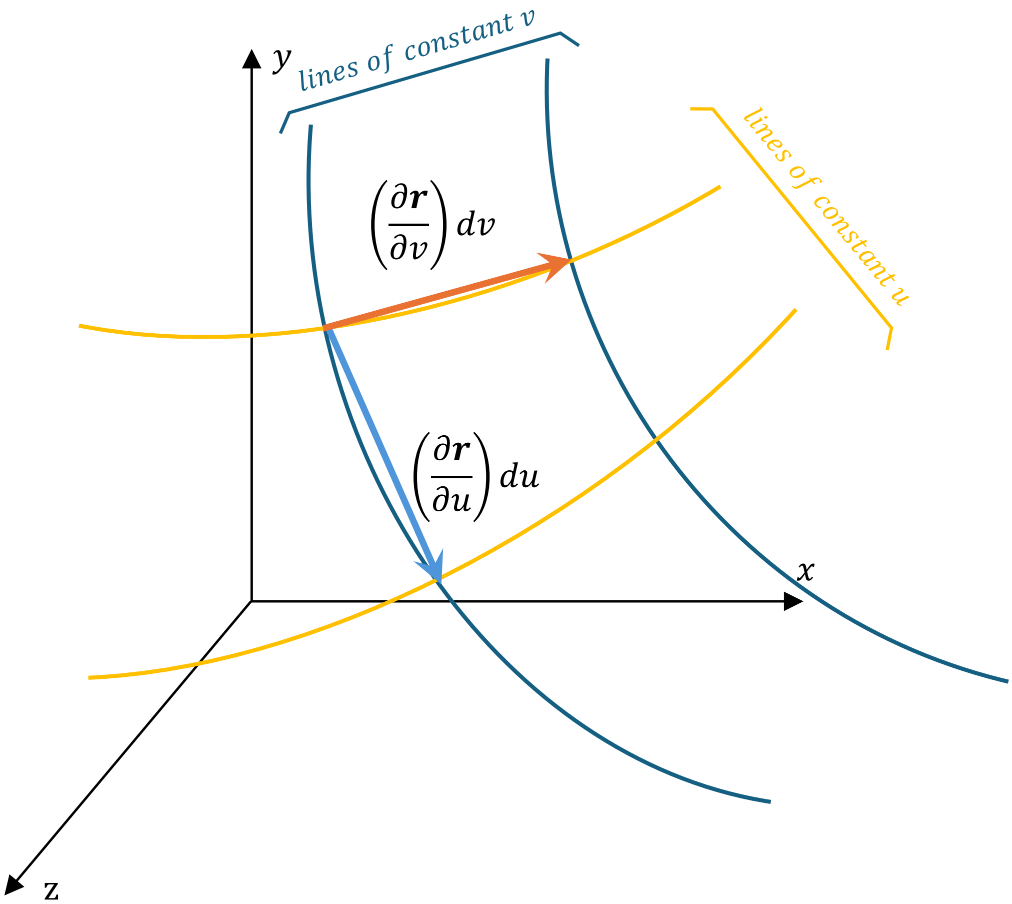


Fig. 0.4.2.1. The tangents to the curves

Similarly, $\frac{\partial \mathbf{r}}{\partial u}$ is not a unit tangent, rather

$$\frac{\partial \mathbf{r}}{\partial u} = h_u \hat{\mathbf{u}} \tag{0.4.2.2}$$

and similarly for $\hat{\mathbf{v}}$. This is exactly what we derived before. Note that the tile is a parallelogram, not a rectangle.

These ideas extent to n-vectors without need for further proof, and in curvilinear with three basis vectors u, v, w, $\mathbf{r} = x(u,v,w)\hat{\mathbf{i}} + y(u,v,w)\hat{\mathbf{j}} + z(u,v,w)\hat{\mathbf{k}}$, we have

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} + h_v dv\hat{\mathbf{v}} + h_w dw\hat{\mathbf{w}} \tag{0.4.2.3a}$$

$$h_u = |\frac{\partial \mathbf{r}}{\partial u}| \qquad h_v = |\frac{\partial \mathbf{r}}{\partial v}| \qquad h_w = |\frac{\partial \mathbf{r}}{\partial w}| \tag{0.4.2.3b}$$

$$|\frac{\partial \mathbf{r}}{\partial u}| = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]^{\frac{1}{2}} \tag{0.4.2.3c}$$

Surface integrals

As we performing surface integral in Cartesians, to find the surface element with normal along $\hat{\mathbf{i}}$ we took a **vector product** of elements in the two orthogonal directions.

$$d\mathbf{S} = (dy\hat{\mathbf{j}}) \times (dz\hat{\mathbf{k}}) = dydz\hat{\mathbf{i}} \tag{0.4.3.1}$$

Clearly we can't apply the same approach in curvilinear coordinates, for the orthogonal and well-behaved length scales nature of Cartesians.

Looking at the elemental parallelogram in u, v, w coordinates, we see that the surface element is planar (not necessarily in the xy-plane).

$$d\mathbf{S} = \frac{\partial r}{\partial u}du \times \frac{\partial r}{\partial v}dv \tag{0.4.3.2a}$$

$$= h_u du\hat{\mathbf{u}} \times h_v dv\hat{\mathbf{v}} \tag{0.4.3.2b}$$

$$= h_u h_v dudv(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \tag{0.4.3.2c}$$

Note that in general curvilinear coordinates, **$\hat{\mathbf{u}} \times \hat{\mathbf{v}}$ doesn't have to equal to $\hat{\mathbf{w}}$** for general curvilinear coordinates **doesn't have to be** orthogonal. However, for an **orthogonal** curvilinear coordinates $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}}$, and

$$d\mathbf{S} = h_u h_v dudv\hat{\mathbf{w}} \tag{0.4.3.3}$$

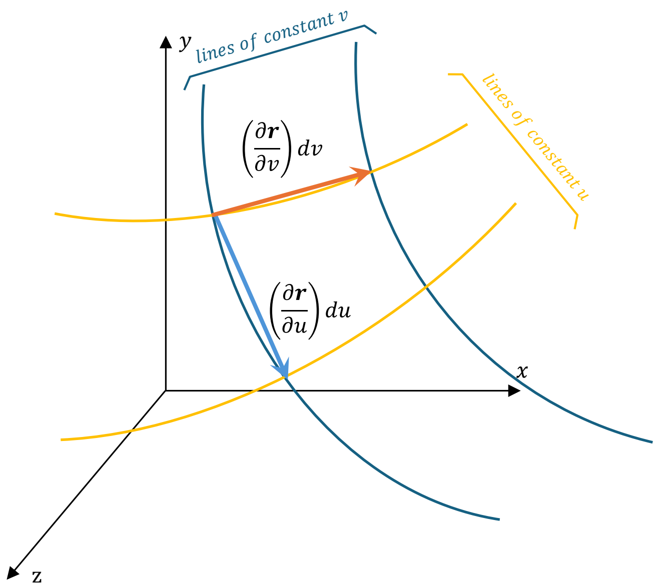


Fig. 0.4.3.1. Surface integrals

A note about Jacobians

Interestingly, if we deal with the change between variables (x, y) and (u, v) in the plane, we arrive at the familiar Jacobian.

$$d\mathbf{S} = dxdy(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) = dxdy\hat{\mathbf{k}} \tag{0.4.3.4a}$$

$$= \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}}\right) \times \left(\frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}}\right) dudv \tag{0.4.3.4b}$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) dudv\hat{\mathbf{k}} \tag{0.4.3.4c}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv\hat{\mathbf{k}} \tag{0.4.3.4d}$$

Usually we just want non-vector integration dS instead of $d\mathbf{S}$, so the vector signs are unimportant. We can see that the *modulus of the Jacobian* can be seen as the area scale factor.

Volume integrals

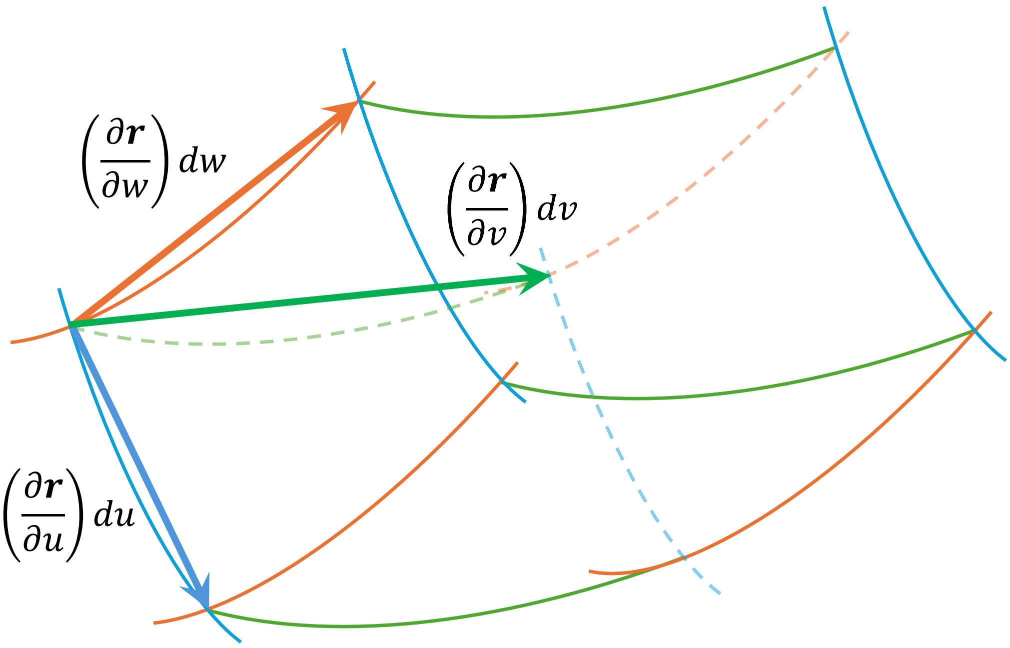


Fig. 0.4.4.1. Volume integrals

Let's transform to curvilinear coordinates u, v, w from x, y, z. It's the volume of a parallelepiped, which we familiar with is given by the *scalar triple product*.

$$dV = \left(\frac{\partial \mathbf{r}}{\partial u}du \times \frac{\partial \mathbf{r}}{\partial v}dv\right) \cdot \frac{\partial \mathbf{r}}{\partial w}dw \tag{0.4.4.1a}$$

$$= h_u h_v h_w dudvdw(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \tag{0.4.4.1b}$$

Recalling that

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}}\right) \tag{0.4.4.2}$$

and combining it with **Eq. 0.4.4.1b**, we get

$$dV = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \cdot \frac{\partial \mathbf{r}}{\partial w} du dv dw \tag{0.4.4.3a}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} du dv dw \tag{0.4.4.3b}$$

the *scalar triple product* is just the Jacobian.

Grad, Div, Curl and ∇^2 in Curvilinear Coordinate Systems

It's possible to obtain general expressions for gard, div and curl in any orthogonal curvilinear coordinate system by making use of the h factors which were introduced in previous sections.

REMINDERS: The unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u} \tag{0.5.1}$$

where $\mathbf{r}$ is the position vector, and

$$h_u = |\frac{\partial \mathbf{r}}{\partial u}| \tag{0.5.2}$$

is the metric coefficient. Similar expressions apply for the other coordinate directions, then

$$d\mathbf{r} = h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw \tag{0.5.3}$$

Grad in curvilinear coordinates

Noting that $U = U(\mathbf{r})$ and $U = U(u, v, w)$, with the properties of the gradient of a scalar field **Eq. 0.2.1.3**, we can easily get

$$\nabla U \cdot d\mathbf{r} = dU = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw \tag{0.5.1.1}$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw \tag{0.5.1.2}$$

Clearly, the only way this can be satisfied for independent du, dv, dw is when

$$\nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial U}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial U}{\partial w}\hat{\mathbf{w}} \tag{0.5.1.3}$$

Divergence in curvilinear coordinates

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear coordinates by making use of the flux property.

We consider an element of volume dV. If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order at a microscale). In previous section, we had **Eq. 0.4.4.1b**

$$dV = h_u h_v h_w du dv dw (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \tag{0.4.4.1b}$$

At this point, for orthogonal curvilinear coordinates, we get

$$dV = h_u h_v h_w du dv dw \tag{0.5.2.1}$$

However, it's not quite a cuboid. The area of two opposite faces will differ as the scale parameters h_u, h_v, h_w are functions of u, v, w in general. That's exactly why there's quite big difference between the divergence in Cartesians and curvilinear coordinates.

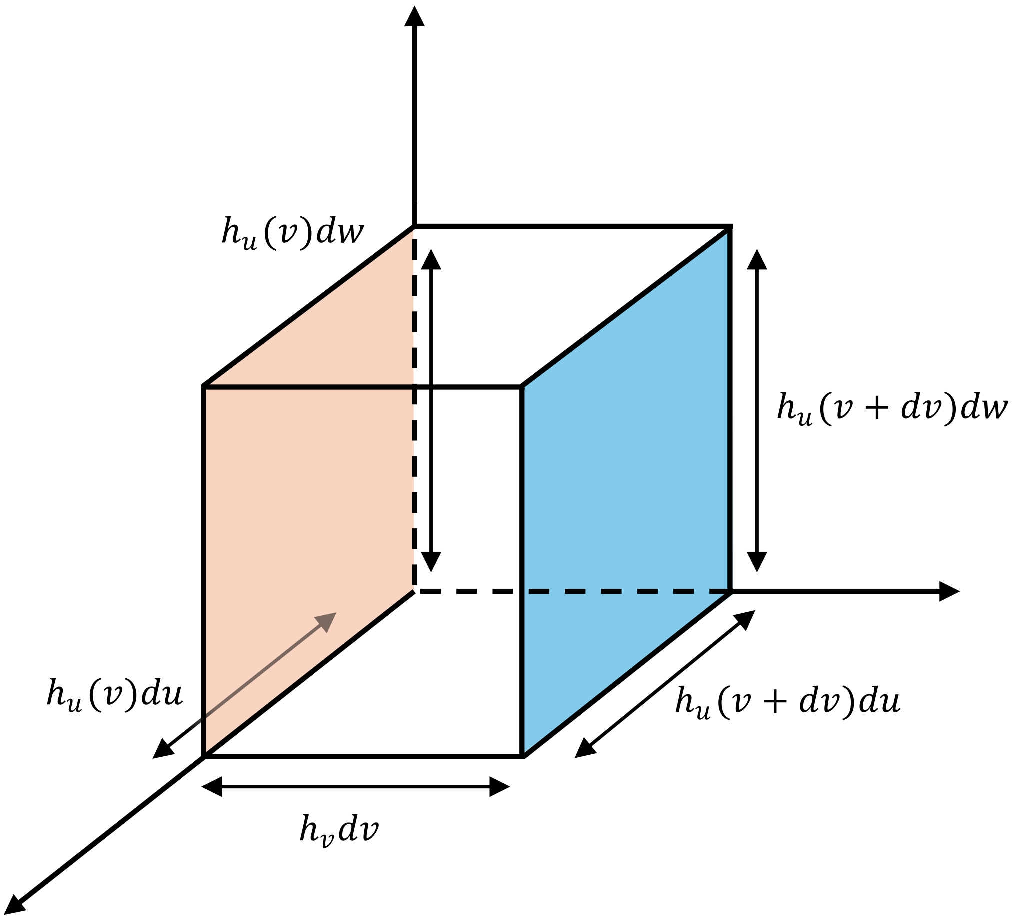


Fig. 0.5.2.1. Elemental volume for calculating divergence in orthogonal curvilinear coordinates

Let us follow the same steps in the previous section. Here we firstly calculate the efflux from the opposite surface in the direction of $\hat{\mathbf{v}}$.

$$= \left[a_v + \frac{\partial a_v}{\partial v}dv\right]\left[h_u + \frac{\partial h_u}{\partial v}dv\right]\left[h_w + \frac{\partial h_w}{\partial v}dv\right]dudw - a_v h_u h_w dudw \tag{0.5.2.2a}$$

$$= \frac{\partial(a_v h_u h_w)}{\partial v}dudvdw \tag{0.5.2.2b}$$

Note that we just multiply the each first term out, and dropped the second and third order terms.

By definition *div* is the net efflux per unit volume, so summing up the other faces

$$div\mathbf{a}\ dV = \left(\frac{\partial(a_u h_v h_w)}{\partial u} + \frac{\partial(a_v h_u h_w)}{\partial v} + \frac{\partial(a_w h_u h_v)}{\partial w}\right)dudvdw \tag{0.5.2.3a}$$

$$\Rightarrow div\mathbf{a}\ h_u h_v h_w\ dudvdw = \left(\frac{\partial(a_u h_v h_w)}{\partial u} + \frac{\partial(a_v h_u h_w)}{\partial v} + \frac{\partial(a_w h_u h_v)}{\partial w}\right)dudvdw \tag{0.5.2.3b}$$

So, finally,

$$div\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u h_v h_w)}{\partial u} + \frac{\partial(a_v h_u h_w)}{\partial v} + \frac{\partial(a_w h_u h_v)}{\partial w}\right) \tag{0.5.2.4}$$

Curl in curvilinear coordinates

Recall from the previous section that, we calculated the z component of curl as the circulation per unit area from **Eq. 0.2.4.4c**

$$dC = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) dxdy \tag{0.2.4.4c}$$

By analogy with our derivation of divergence, you will realize that the scale parameters h_u, h_v, h_w will again disturb us. The opposite sides are no longer quite of the same length. The lower of the pair in **Fig. 0.5.3.1** is of length $h_u(v)du$, while the upper is of length $h_u(v + dv)du$.

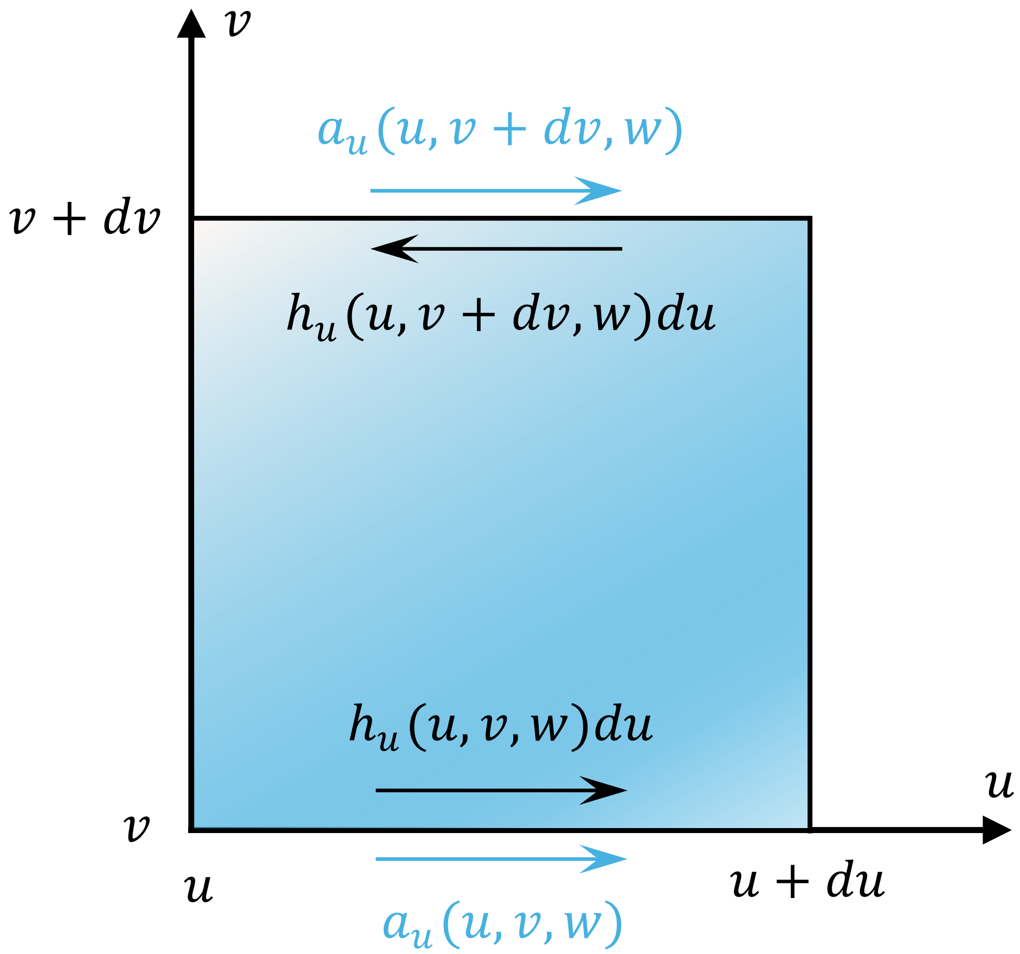


Fig. 0.5.3.1. Elemental loop for calculating curl in orthogonal curvilinear coordinates

Summing this pair **counterclockwise** gives a contribution to the circulation

$$a_u(v)h_u(v)du - a_u(v + dv)h_u(v + dv)du = -\frac{\partial(h_u a_u)}{\partial v} dvdu \tag{0.5.3.1}$$

and together with the horizontal pair, we get

$$dC = \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right) dudv \tag{0.5.3.2}$$

Now we must notice what we considered is an element of surface dS. By analogy with our derivation of divergence, you will clearly realize that if the curvilinear coordinates are orthogonal, then the little surface is a rectangle (to first order at a microscale). In previous section, we had **Eq. 0.4.3.2c**

$$d\mathbf{S} = h_u h_v du dv (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \tag{0.4.3.2c}$$

For orthogonal curvilinear coordinates, we get **Eq. 0.4.3.3**

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{w}} \tag{0.4.3.3}$$

Here we just care about the non-vector integration dS, so we can write the area of the unit surface as

$$dS = h_u h_v du dv \tag{0.5.3.3}$$

Combining the **Eq. 0.5.3.3** with **Eq. 0.5.3.2**, we get the circulation per unit area

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right) \tag{0.5.3.4}$$

The circulation per unit area is of the direction $\hat{\mathbf{w}}$. Summing the circulation of the other two orientations $\hat{\mathbf{u}}, \hat{\mathbf{v}}$, we finally get

$$\begin{aligned} curl\mathbf{a}(u,v,w) =& \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} + \\ & \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} + \\ & \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}} \end{aligned} \tag{0.5.3.5}$$

Usually we want it to be *determinant form* just like in Cartesians

$$curl\mathbf{a}(u,v,w) = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix} \tag{0.5.3.6}$$

Laplacian in curvilinear coordinates

With the **Eq. 0.5.1.3** and **Eq. 0.5.2.4**, substitution of the components of $gradU$ into the expression for $div\mathbf{a}$ immediately gives the following expression for the *Laplacian* in general orthogonal coordinates

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right] \tag{0.5.4.1}$$

Grad, Div, Curl and ∇^2 in cylindrical polars

REMINDERS: Here $(u, v, w) \to (r, \phi, z)$. The position vector is $\mathbf{r} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right|$, etc.

$$\Rightarrow h_r = \sqrt{(\cos^2\phi + \sin^2\phi)} = 1 \tag{0.5.5.1a}$$

$$h_\phi = \sqrt{(r^2\cos^2\phi + r^2\sin^2\phi)} = r \tag{0.5.5.1b}$$

$$h_z = 1 \tag{0.5.5.1c}$$

$$\Rightarrow gradU = \frac{\partial U}{\partial u}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \tag{0.5.5.2a}$$

$$div\mathbf{a} = \frac{1}{r}\left(\frac{\partial(ra_r)}{\partial r} + \frac{\partial a_\phi}{\partial \phi}\right) + \frac{\partial a_z}{\partial z} \tag{0.5.5.2b}$$

$$curl\mathbf{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial(ra_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\hat{\mathbf{k}} \tag{0.5.5.2c}$$

$$\nabla^2 U = \frac{1}{r}\left(\frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \tag{0.5.5.2d}$$

Grad, Div, Curl and ∇^2 in spherical polars

REMINDERS: Here $(u, v, w) \to (r, \theta, \phi)$. The position vector is $\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}$, and $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right|$, etc.

$$\Rightarrow h_r = \sqrt{\left[\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta\right]} = 1 \tag{0.5.6.1a}$$

$$h_\theta = \sqrt{\left[r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta\right]} = r \tag{0.5.6.1b}$$

$$h_\phi = \sqrt{r^2\sin^2\theta(r^2\cos^2\phi + r^2\sin^2\phi)} = r\sin\theta \tag{0.5.6.1c}$$

$$\Rightarrow gradU = \frac{\partial U}{\partial u}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{0.5.6.2a}$$

$$div\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi} \tag{0.5.6.2b}$$

$$curl\mathbf{a} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\phi)\right] + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left[\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right] \tag{0.5.6.2c}$$

$$\nabla^2 U = \frac{1}{r^2}\left(2r\frac{\partial U}{\partial r} + r^2\frac{\partial^2 U}{\partial r^2}\right) + \frac{1}{r^2 sin\theta}\left(\cos\theta\frac{\partial U}{\partial \theta} + \sin\theta\frac{\partial^2 U}{\partial \theta^2}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} \tag{0.5.6.2d}$$

♣ Examples

Question 1: Find *curl* $\mathbf{a}$ in (i) Cartesians and (ii) spherical polars when $\mathbf{a} = x(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$.

A1:

(i) In Cartesians

$$curl\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\mathbf{j}} + y\hat{\mathbf{k}} \tag{0.5.7.1}$$

(ii) In spherical polars, $x = r\sin\theta\cos\phi$ and $\mathbf{r} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$. So

$$\mathbf{a} = r^2 \sin\theta\cos\phi\hat{\mathbf{r}} \tag{0.5.7.2a}$$

$$\Rightarrow a_r = r^2\sin\theta\cos\phi \qquad a_\theta = 0 \qquad a_\phi = 0 \tag{0.5.7.2b}$$

Substituting **Eq. 0.5.7.2b** into **Eq. 0.5.6.2c**, we get

$$\begin{aligned} curl\mathbf{a} &= \cdots \\ &= \hat{\boldsymbol{\theta}}(-r\sin\phi) + \hat{\boldsymbol{\phi}}(-r\cos\theta\cos\phi) \end{aligned} \tag{0.5.7.3}$$

With the rotation matrix $[R]$ from Cartesians to spherical polars, we can check these two results should be the same.

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} = [R] \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \tag{0.5.7.4}$$

Subtituting **Eq. 0.5.7.4** into **Eq. 0.5.7.3**, we get that the result in spherical polars is

$$curl\mathbf{a} = \cdots$$

$$= -r\cos\theta\hat{\mathbf{j}} + r\sin\theta\sin\phi\hat{\mathbf{k}} \tag{0.5.7.5a}$$

$$= -z\hat{\mathbf{j}} + y\hat{\mathbf{k}} \tag{0.5.7.5b}$$

which is exactly the result in Cartesians.

Question 2: Find the *divergence* of the vector field $\mathbf{a} = r\mathbf{c}$ where $\mathbf{c}$ is a constant vector (i) using Cartesian coordinates and (ii) using spherical polar coordinates.

A2:

(i) Using Cartesian coordinates

$$\begin{aligned} div\mathbf{a} &= \cdots \\ &= \frac{1}{r}\mathbf{r}\cdot\mathbf{a} \end{aligned} \tag{0.5.7.6}$$

(*ii*) Using spherical polar coordinates

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}} \tag{0.5.7.7}$$

Note that we don't have the expressions of a_r, a_θ, a_ϕ, but we know that it's the same point in space whatever the coordinate system, so

$$a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}} \tag{0.5.7.8}$$

Combining the **Eq. 0.5.7.8** with **Eq. 0.5.7.4**, we get

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \tag{0.5.7.9a}$$

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} \tag{0.5.7.9b}$$

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \tag{0.5.7.9c}$$

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top [R]^\top \tag{0.5.7.9d}$$

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = [R] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{0.5.7.9e}$$

For our particular problem, $a_x = rc_x$, etc, where c_x is a constant, so now we can write

$$a_r = r(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z) \tag{0.5.7.10a}$$

$$a_\theta = r(\cos\theta\cos\phi c_x + \cos\theta\sin\phi c_y - \sin\theta c_z) \tag{0.5.7.10b}$$

$$a_\phi = r(-\sin\phi c_x + \cos\phi c_y) \tag{0.5.7.10c}$$

Here we got the expressions already. Now all we need to do is to substitute these into **Eq. 0.5.6.2b**

$$\begin{aligned} div\mathbf{a} =& 3(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z) + \\ & \frac{1}{\sin\theta}\left[(\cos^2\theta - \sin^2\theta)(\cos\phi c_x + \sin\phi c_y) - 2\sin\theta\cos\theta c_z\right] + \\ & \frac{1}{\sin\theta}(-\cos\phi c_x - \sin\phi c_y) \end{aligned} \tag{0.5.7.11}$$

A bit more bashing and you will find

$$\begin{aligned} div\mathbf{a} &= \sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z && (0.5.7.12a) \\ &= \hat{\mathbf{r}}\cdot\mathbf{a} && (0.5.7.12b) \end{aligned}$$

That's **exactly** what we worked out using Cartesian coordinates, **Eq. 0.5.7.6**.

Through the above example exercises, I will summarize some interesting points:

- Just as physical vectors are independent of their coordinate systems, so are **differential operators**.
- **Rotation matrices** are useful for changing the coordinate systems.
- Spherical polar coordinates **are not always** a good coordinate system.

Chapter 1 Introduction to Electrostatics

1.1 Guss

$$e^{j\pi} + 1 = 0 \tag{1.1}$$

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1.2 Summary

The summary

Here is the summary. Referring to section 1.1 on page 1

References

[1] EuphoricRhino. Jackson E-M Notes, 2023.
`https://www.zhihu.com/people/ni-xu-wei-13/posts?page=6`.